S.-T. Yau College Student Mathematics Contests 2017

Oral Exam of Geometry and Topology

Team Problems

1. (a) Describe the loop space ΩS^2 and path space PS^2 of the sphere S^2 in the following fibration:

$$\begin{tikzcd} \Omega S^2 \arrow[r] & PS^2 \arrow[d] \\ & S^2. \end{tikzcd}$$

(b) Compute the cohomology of the loop space ΩS^2. What is the ring structure of $H^*(\Omega S^2)$?

2 (Synge theorem)**.** Let M be an even-dimensional compact Riemannian manifold with positive sectional curvature.

(a) When M is orientable, show that M is simply connected.

(b) When M is unorientable, what is $\pi_1(M)$?

3. (a) Let C be a smooth curve on the sphere. The Crofton formula expresses the arc length $L(C)$ of the curve C as

$$L(C) = \frac{1}{4}\int_{S^2} n(C \cap W^{\perp})dW.$$

Here $W^{\perp}$ is the plane with normal W going through the origin and $n(C \cap W^{\perp})$ is the number of points in the intersection of C and $W^{\perp}$.

(b) Sketch a proof of Crofton formula.

4. Let $\Omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ and $\alpha > 0$. Suppose D is the surface in $\mathbb{R}^3$ defined by $\{(x, y, z) \,|\, x^2 + y^2 + z^2 = 1, z \geq \alpha\sqrt{x^2 + y^2}\}$.

(a) Show that $\Omega|_D$ is an orientation form and makes D an oriented manifold with boundary.

(b) Evaluate $\int_D \Omega$. Your answer should be in terms of α.